General relativity reformed to a genuine Yang-Mills gauge theory for gravity

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Abstract

The theory of general relativity is reformed to a genuine Yang-Mills gauge theory of the Poincaré group for gravity. Several pathologies of the conventional theory are thus removed, but not every GR vacuum satisfies the Y-M equations. The sector of GR solutions which survive is fully classified and it is found to include the Schwarzschild black hole. Two other solutions presented here have no GR counterpart and they describe expanding Friedmann universes with torsion which vanishes only asymptotically. They are discussed along with novel theoretical possibilities, such as a well-defined energy-momentum tensor for the gravitational field, and novel perspectives for unification and quantization.

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1 Introduction

It is quite amazing that shortly after the final formulation of Einstein's theory of general relativity, Elie Cartan had essentially supplied the geometric framework for a genuine Yang-Mills gauge theory of gravity (GGG)[1]. Following the early and subsequent attempts (cf. [2],[3] for reviews) the existence of a GGG, notably one based on the Poincaré group $\mathcal{P} = ISO(1,3)$, has rather been taken for granted [3][4]. As far as the present author is aware, no actual example or model for a GGG has been given, a fact related not only to objective difficulties, but apparently also to certain misconceptions which erroneously attribute gauge-theoretic aspects to GR and vice-versa. Here, we will particularly need to clarify and diggress in the following: (i) The nature of gauge, isometry and holonomy transformations, after the gauge fields have been soldered onto the space they inhabit. (ii) The relation of these transformations to what is actually lost by the (usual) supression of the translational degrees of freedom when gauging $\mathcal{P}$. (iii) The identification of the quadratic-in-cuvature part of the action which remains a topological invariant when the torsion $\mathcal{T}$ does not vanish, and the emergence of linear-in-$\mathcal{R}$ and quadratic in $\mathcal{T}$ terms. (iv) The reform of GR as an acceptable GGG.

The above also outline our motivation and our approach for the genuine (meaning: straigt-forward, real, unsuppressed) Y-M gauge theory of $\mathcal{P}$, presented in sections 2,3 and shown to be free of certain pathologies of GR. In section 4, the sector of GR vacua which survive (namely satisfy the Y-M equations) is fully classified and shown to include the Schwarzschild space-time, and thus the observational backing of the latter. Two other solutions found describe expanding Friedmann universes and have no GR counterparts. They are discussed in section 5, along with the rest of our findings. Our notation and terminology generally follows that of [4].

2 Gauging the Poincaré group

The generators $\{P_a, M_{aa'}\}$ of $\mathcal{P}$ satisfy the well known commutation relations

$$[P_a, P_b] = 0, \quad [M_{aa'}, P_b] = 2\delta^c_{[a} g_{a']b} P_c, \quad [M_{aa'}, M_{bb'}] = 4\delta^c_{[a} g_{a'][b} \delta^{c'}_{b']} M_{cc'}. \tag{1}$$

They are assigned, respectively, to translations and Lorentz $SO(1,3)$ rotations in Minkowski space-time $M^4_\circ$ serving as a representation space. $\mathcal{P}$ acts on $M^4_\circ$ as a maximal group of isometries which preserve the standard flat (Minkowski) metric

$$g_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{2}$$

The 1-form valued potential

$$\mathcal{A} = \theta^a P_a + \frac{1}{2}\omega^{ab} M_{ab}, \tag{3}$$

inhabits as a gauge field some target space-time M^4, whereupon soldered defines the vierbein θ^a and connection ω^{ab} structure of M^4. This construction will be discussed in more detail shortly. For the moment we note that the field strength $\mathcal{F}$ and the covariant derivative $\mathcal{D}$ are defined in terms of $\mathcal{A}$ as usual

$$\mathcal{T}^a P_a + \frac{1}{2}\mathcal{R}^{ab} M_{ab} = \mathcal{F} := d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}], \quad \mathcal{D}\mathcal{F} := d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0. \tag{4}$$

The suggestive identifications for the components of $\mathcal{F}$ reflect the emergence of precisely Cartan's structure equations for the torsion and curvature 2-forms as

$$\mathcal{T}^a := D\theta^a := d\theta^a + \omega^a_{\cdot b} \wedge \theta^b, \qquad \mathcal{R}^a_{\cdot b} := d\omega^a_{\cdot b} + \omega^a_{\cdot c} \wedge \omega^c_{\cdot b}, \tag{5}$$

which automatically satisfy the Bianchi identities

$$D\mathcal{T}^a = \mathcal{R}^a_{\cdot b} \wedge \theta^b, \qquad D\mathcal{R}^a_b = 0. \tag{6}$$

The Y-M action is

$$\mathcal{I}_{YM} = \int_{M^4} \mathrm{tr}\,(\mathcal{F} \wedge *\mathcal{F}) = \frac{1}{2}\int_{M^4} \mathcal{R}^{ab} \wedge *\mathcal{R}_{ab} = \frac{1}{4}\int_{M^4} R^{abcd} R_{abcd}\, \eta, \tag{7}$$

where the $*$ duality is with respect to the metric (2), the functions R^a_{bcd} specify the components of $\mathcal{R}^a_b$ in the θ^a frame, and η is the invariant volume 4-form. The Bianchi identity and (from variation of $\mathcal{A}$) the vacuum Y-M equations are

$$\mathcal{D}\mathcal{F} = 0, \qquad \mathcal{D} * \mathcal{F} = 0. \tag{8}$$

Of these, the first is precisely eqivalent to the Bianchi identities (6), while the second may be written in tems of the components of $\mathcal{F}$ as

$$D * \mathcal{T}^a = *\mathcal{R}^a_{\cdot b} \wedge \theta^b, \qquad D * \mathcal{R}^a_b = 0. \tag{9}$$

These field equations may be integrated to provide vacuum configurations according to prescribed boundary or asymptotic conditions. One may also look for (anti-)self-dual solutions, namely with

$$\mathcal{F} = \pm * \mathcal{F} \qquad \leftrightarrow \qquad \mathcal{T}^a = \pm * \mathcal{T}^a, \;\; \mathcal{R}_{ab} = \pm * \mathcal{R}_{ab}, \tag{10}$$

which, however, must also satisfy a set of constraint equations (the analogue of Einstein's equations - cf., (17) or (18) in the next section).

The obvious question now is whether the above can be *really* considered as a GGG, after the vierbein θ^a has been soldered on the target space. Relatedly, the following arguments have appeared in the literature (the ennumeration is in correspondence to the one in section 1): (i) When the vierbein θ^a is soldered onto M^4, our freedom to translate is lost. (ii) Torsion is automatically absent from the Y-M action (the $\mathcal{T}\mathcal{T}$ and $\mathcal{T}\mathcal{R}$ contributions are easily seen to drop out, as shown in (7)). (iii) The absense of a linear-in-$\mathcal{R}$ term in (7) combined with the observation that the $\mathcal{R}\mathcal{R}$ term is a topological invariant (hence with no contribution to the classical theory) rules out reducibility to GR. (iv) In view of these (i-iii) impasses, the obvious resolution is to suppress translations, set $\mathcal{T} = 0$ (or get the same result from a variation of the Hilbert-Einstein action), and utilize the action (7) only in the context of quantum gravity.

We will attempt a closer examination to see that these arguments are misleading or just false, and then proceed to complete our basic results in the next section.

3 Young-Mills action, field equations and constraints

Still keeping the correspondence with the ennumeration (i-iv) in sections 1,2, we observe the following.

(i) After soldering we indeed do not have the freedom to translate isometrically (or even rotate likewise), but all that is in M^4. The point is not to confuse isometries (and the other transformations mentioned) in M^4 and in $M^4_\circ$. To better see this, one must realize that although the metric on M^4 is identical to the metric (2) of $M^4_\circ$, the two space-times are only *locally* identical (except for any isolated singular points in M^4), with the differences prescribed by the local gauge potential $\mathcal{A}$. Thus, any given $\mathcal{A}$ fixes the geometry of the target space to a particular $M^4\left(\mathcal{A}\right)$ configuration, generally with no symmetry at all, so that the soldered vierbein θ^a will necessarily be non-holonomic and the connection ω^{ab} non-integrable. M^4 is not even gauge invariant, because a gauge transformed $\mathcal{A}'$ will generally define a different geometry $M^4\left(\mathcal{A}'\right)$. What *is* gauge invariant, is the Y-M action (7).

(ii) GR is not a Y-M gauge theory, so it makes perfect sense to set there $\mathcal{T}=0$. We may certainly also have configurations with $\mathcal{T}=0$ in the Y-M gauge theory of $\mathcal{P}$, but having *a priori* $\mathcal{T}=0$ (equivalently, suppressing translations), simply means that we are dealing with the gauge theory of $SO(1,3)$, rather than $ISO(1,3)$. In a gauge-theoretic context, even if one started with $SO(1,3)$ as the gauge group, one would have to enlarge it to $ISO(1,3)$ (and it is in this context that one can better appreciate Cartan's major contribution and foresight on the subject). The reason is that the representation space of $SO(1,3)$, namely $M^4_\circ$, is independently also involved as locally identical to M^4 not in any gauge-theoretic context, but from the fundamental definition of M^4 as a pseudo-Riemannian manifold. The nessecity to enlarge $SO(1,3)$ to $ISO(1,3)=\mathcal{P}$ as a gauge group now follows from the identification of $\mathcal{P}$ and the trivial group as the isometry and holonomy groups *in that order* for $M^4_\circ$ and *in reverse order* at the other extreme of a general M^4. As a result, the previous local identification between $M^4_\circ$ and M^4 can be simultaneously also established in the gauge-theoretic context as well. This deep and elegant result would be lost (and GR could not be reformed to a GGG) if the torsion were identically zero.

(iii) When the torsion is not identically zero, the action (7) *is not* a topological invariant, but it is related to the Euler characteristic of M^4, as we will see shortly. As a result, variation of (7) with respect to g_{ab} makes perfect sense. In fact, it supplies a well-defined energy-momentum tensor for the gravitational field, as well as constraints which may be viewed as the analogue of Einstein's equations.

(iv) With the earlier impasses out of the way, we may now proceed to complete our basic results.

Let $\overset{\circ}{\omega}_{ab}$, $\overset{\circ}{D}$, $\overset{\circ}{\mathcal{R}}_{ab}$, $\mathcal{K}_{ab}$, denote, in that order, the Cristoffel part of the connection $\omega_{ab}=\overset{\circ}{\omega}_{ab}-\mathcal{K}_{ab}$, the Cristoffel covariant derivative, the associated curvature 2-form, and the contorsion 1-form. Then, we may re-write (5) as

$$\mathcal{T}^a = D\theta^a = d\theta^a + \left(\overset{\circ}{\omega}{}^a_{\cdot b} - \mathcal{K}^a_{\cdot b}\right)\wedge\theta^b = -\mathcal{K}^a_{\cdot b}\wedge\theta^b, \quad \mathcal{R}_{ab} = \overset{\circ}{R}_{ab} - \mathcal{H}_{ab}, \tag{11}$$

where we have defined

$$\mathcal{H}^a_{\cdot b} := \overset{\circ}{D}\,\mathcal{K}^a_{\cdot b} - \mathcal{K}^a_{\cdot c}\wedge\mathcal{K}^c_{\cdot b}. \tag{12}$$

The action (7) may now equivalently be written as

$$\mathcal{I}_{YM} = \frac{1}{2}\int_{M^4} \mathcal{R}^{ab} \wedge *\mathcal{R}_{ab} = \int_{M^4} \frac{1}{2}\overset{\circ}{\mathcal{R}}{}^{ab} \wedge * \overset{\circ}{\mathcal{R}}_{ab} - \overset{\circ}{\mathcal{R}}{}^{ab} \wedge *\mathcal{H}_{ab} + \frac{1}{2}\mathcal{H}^{ab} \wedge *\mathcal{H}_{ab}. \tag{13}$$

The first term on the rhs gives the Euler characteristic of M^4, and we also observe the emergence of the linear in $\mathcal{R}$ and quadratic in $\mathcal{T}$ terms we referred to earlier.

The energy-momentum tensor of the $\mathcal{F}$ field is determined by variation of (13) with respect to the metric g_{ab} as equal to

$$\begin{aligned} E_{ab} : \; &= \frac{1}{2}\left(R_{pqra}R^{pqr}_{\;\;\cdots\, b} - \frac{1}{4}R_{pqrs}R^{pqrs}g_{ab}\right) \\ &= \frac{1}{2}\left(H_{pqra}H^{pqr}_{\;\;\cdots\, b} - \frac{1}{4}H_{pqrs}H^{pqrs}g_{ab}\right) - \left(\overset{\circ}{R}_{pqr(a}\, H^{pqr}_{\;\;\cdots\, b)} - \frac{1}{4}\,\overset{\circ}{R}_{pqrs}\, H^{pqrs}g_{ab}\right), \end{aligned} \tag{14}$$

where $R_{\cdots\cdot}$, $\overset{\circ}{R}_{\cdots\cdot}$,$H_{\cdots\cdot}$ are the components of $\mathcal{R}_{\cdot\cdot}$, $\overset{\circ}{\mathcal{R}}_{\cdot\cdot}$, $\mathcal{H}_{\cdot\cdot}$ in the θ^a frame. We observe that there are two main contributions on rhs of (14), with the first one clearly identifiable as the energy-momentum tensor of the $\mathcal{H}$ field

$$E^{(\mathcal{H})}_{ab} := \frac{1}{2}\left(H_{pqra}H^{pqr}_{\;\;\cdots\, b} - \frac{1}{4}H_{pqrs}H^{pqrs}g_{ab}\right). \tag{15}$$

Even if the this $\mathcal{H}$ field tuned out to be partly or entirely of non-gravitational nature, we could subtract from (14) that non-gravitational contribution, to again end up with a well-defined energy-momentum tensor for the gravitational field (cf. also discussion in the last section). In any case, variation of the overall action with respect to the metric should vanish [2]. Thus, if an external source or field were added to the action (7), variation with respect to g_{ab} would give

$$E_{ab} + E^{(\mathrm{Sources})}_{ab} = 0. \tag{16}$$

This set of ten constraints establishes the energy-momentum balance of the theory and may be viewed as the analogue of Einstein's equations in conventional GR, equivalently written as

$$\overset{\circ}{R}_{pqr(a}\, H^{pqr}_{\;\;\cdots\, b)} - \frac{1}{4}\,\overset{\circ}{R}_{pqrs}\, H^{pqrs}g_{ab} = E^{(\mathcal{H})}_{ab} + E^{(\mathrm{Sources})}_{ab}. \tag{17}$$

We observe that these equations involve the entire Riemann tensor (namely not just its contraction to the Ricci tensor as in conventional GR), hence they include an explicit and non-trivial contribution from the Weyl tensor. To summarize our results, the vacuum equations for our GGG are

$$D * \mathcal{T}^a = *\mathcal{R}^a_{\cdot b} \wedge \theta^b, \quad D * \mathcal{R}^a_b = 0, \quad E_{ab} = 0. \tag{18}$$

The last equation does not imply zero field strength, because $\mathcal{P}$ is non-compact. This will be further discussed and seen explicity in the examples of the next section, to which we now turn.

4 Solutions and simple vacuum configurations

There is an overlap between GR vacua and solutions to (18), which obviouly belong to the $\mathcal{T} = 0$ sector of the latter. One can verify that all these solutions, common to the two theories, fall in either of the following two classes.

Class A: All GR vacua with self-dual or anti-self-dual Riemann tensor.

Class B: The rest of GR vacua, whose curvature satisfies $D * \mathcal{R}^a_b = 0$.

One can prove that the Schwarzschild black hole of GR actually belongs to class B. This is an important result, because it establishes for the present theory identical Newtonian and post-Newtonian limits with GR, as far as they are drawn from the Schwarzschild geometry.

We will now present and examine two other solutions which do not belong to either of the above classes. We search for cosmological solutions to (18), with spatial homogeneity and isotropy ('Friedmann models'). This means that we may profitably choose the frame θ^a with $\theta^0 = dt$, $\theta^i = a(t)\sigma^i$ $(i, j, \ldots = 1, 2, 3)$), and with the radius (expansion scale) a depending only on the cosmic time t as indicated. The geometry of the spatial sections is fixed by the relation $d\sigma^i = \frac{k}{2}\epsilon^i_{jk}\sigma^j \wedge \sigma^j$ as open (giving flat R^3 sections) or closed (giving round S^3 sections), depending on the value of the parameter $k = 0, 1$.

In view of the homogeneity and isotropy, the torsion cannot define any direction in space, so it may be choosen as

$$\mathcal{T}^a = \theta^a \wedge d\phi, \qquad \leftrightarrow \qquad \mathcal{K}^{ab} = \phi^a\theta^b - \phi^b\theta^a, \tag{19}$$

where ϕ is a scalar field depending only on t. The system of eqs (18) now admits solutions which we have determined as

$$a = \frac{1}{2}\, t\, e^{\phi}, \quad e^{\phi} = \sqrt{\left(\frac{t}{t_0}\right)^2 - k^2}, \tag{20}$$

where t_0 is a constant. For both $k = 0, 1$ values, these solutions represent ever-expanding universes with an initial singularity. The expantion scale increases exactly or asymptotically as t^2 and the torsion goes to zero asymptotically as $1/t$ in the θ^a frame, as seen from (19). These solutions have no GR counterpart: if the torsion is identically zero, then, for $k = 0$ the expansion is lost and we only have Minkowski space-time as a solution to (18), while for $k = 1$ there is no solution at all.

5 Discussion and conclusions

We have presented here what is essentially a new theory of gravity, based on the Y-M gauging of the Poincaré group $\mathcal{P}$. Although clearly distinct from GR, it may be viewed as a reform thereof, because it shares with the latter its zero-torsion sector of solutions. These we have exhaustively categorized in the A and B classes defined in section 4. As seen, the Schwarzschild black hole is a solution in class B. In the same section, we gave two other solutions, which are ever-expanding Friedmann models with an initial singularity, asymptotically zero torsion, and no GR counterpart or limit. As mentioned, the present theory shares the same observational backing, and simultaneously appears to be free of certain pathologies of GR, as itemized below.

(i) The number of $4\times4+6\times4+10=50$ *a priori* independent field equations (18) is precisely equal to the number of degrees of freedom for the likewise independent variables θ^a,ω_{ab},g_{ab}. For the same variables, GR provides an *under-determined* system of 10 field equations plus 24 coming from the zero-torsion constraint [4]. We have also seen that the constraint equations (17) essentially replace the conventional set of Einstein's euations, so that (unlike the case in GR), there is an explicit and non-trivial contribution from the Weyl tensor.

(ii) The presence of torsion allows the elegant inter-relation between the gauge, isometry and general holonomy groups, as outlined under (ii) in section 3.

(iii) There is a well-defined energy-momentum tensor for the gravitational field. In particular, as seen from (14), E_{ab} is equal to zero for all solutions in the A or B class, in spite of the fact that the field strength in the corresponding vaccua does not vanish (due to the non-compactness of $\mathcal{P}$, as mentioned earlier). The same configurations viewed as GR vacua are known to have generally ill-defined or non-existent energy-momentum tensor [4].

(iv) The unification and quantization aspects of a Y-M formulation are generally applicable in the present case, as long as they do not require compactness of the gauge group (the last qualification *could* of course re-induce some of the serious impasses associated with gravity, but this is *not a priori* obvious [6]).

Expanding very briefly on the above (and perhapse also on the side of speculation), we note, on the one hand, the apparent association of ω_{ab} (or $\overset{\circ}{\omega}_{ab}$) with what should be cosidered as the gravitational bosons. At the same time, with the linear-in-$\mathcal{R}$ contribution in the action (13) viewed as a generalization of the Hilbert-Einstein Lagrangian, it is clear from (17) that the generally non-constant gravitational coupling is inherently specified, e.g., as function of ϕ in the simpler models. In a closely related 'Machian' behavior, such couplings have been quite generally shown to be positive and asymptotically constant [7]. On the other hand, the physical identification of the vierbein θ^a as a gauge field remains unclear, obviously dependent on the physical identification of the tosrsion and consequently of the $\mathcal{H}$ field in the action (13). If not of gravitational nature, the $\mathcal{H}$ field could be the carrier of some other fundamental interaction (e.g., as exemplified by the ϕ field in the simple models of section 4). In that context, the electroweak interaction appears to be a plaussible candidate (e.g., as attempted in recent work [8]). In any case, torsion is here a dynamical ('propagating') field, so its direct association with spin [2] is ruled out.

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